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ORIGINAL PAPER

Sufficient global optimality conditions for weakly convex minimization problems

Z. Y. Wu

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Abstract In this paper, we present sufficient global optimality conditions for weakly convex minimization problems using abstract convex analysis theory. By introducing (L, X)-subdifferentials of weakly convex functions using a class of quadratic functions, we first obtain some sufficient conditions for global optimization problems with weakly convex objective functions and weakly convex inequality and equality constraints. Some sufficient optimality conditions for problems with additional box constraints and bivalent constraints are then derived.

Keywords Global optimization · Optimality conditions · Weakly convex minimization

AMS Subject Classification 41A65 · 41A29 · 90C30

1 Introduction

Sufficient optimality conditions in global optimization for some special kinds of nonconvex optimization problems have been studied by many researchers (see for example [1–6,10] and references therein). Recently, a new approach for establishing sufficient conditions was suggested in Refs. [7,8,12,15]. This approach is based on abstract convex analysis (see, for e.g., [9,11,13]) as (L, X)-subdifferential and L-normal cone, where L is a set of real valued functions defined on $\mathbb{R}^n$, $X \subset \mathbb{R}^n$.

Z. Y. Wu (✉)
Department of Mathematics, Chongqing Normal University, Chongqing 400047, People's Republic of China

Present address:
Z. Y. Wu
School of Information Technology and Mathematical Sciences, University of Ballarat, Ballarat, 3353 VIC, Australia
e-mail: z.wu@ballarat.edu.au

It was shown in Refs. [7,8,15] that (L,X)-subdifferential with respect to certain sets of quadratic functions can be successfully applied to derive sufficient global optimality conditions for nonconvex problems with a quadratic objective function subject to quadratic constraints and/or box constraints and bivalent constraints.

In this paper, we extend the approach based on abstract convexity for examination of a large class of optimization problems with weakly convex functions involved (see Definition 2.1). The class of weakly convex functions is very large: an arbitrary C^2 nonconvex function defined on a compact set is a weakly convex function. In this paper, we study the global optimality conditions for optimization problems with weakly convex functions involved using (L,X)-subdifferentials where L is the following set of quadratic functions:

$$L = \{l : l(x) = \alpha\|x\|^2 + x^T\beta,\ \alpha \in \mathbb{R},\ \beta \in \mathbb{R}^n\}.$$

Let H be the set of L-affine functions $h(x) = l(x) + c,\ l \in L, c \in \mathbb{R}$. Abstract convexity with respect to the set H has been studied by many authors (see [9,11] and references therein).

The layout of the rest of the paper is as follows. Section 2 presents the notions of (L,X)-subdifferentials, L-normal cones, and weakly convex functions. Sufficient conditions for a class of nonconvex minimization problems are presented in Sect. 3. This section contains also description of (L,X)-subdifferentials and sufficient conditions for global minimizers of weakly convex problems. Section 4 derives optimality conditions for some special cases of weakly convex minimization problems.

2 Preliminaries

In this section, we present basic definitions that will be used throughout the paper. We use the following notation: $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$, $\mathbb{R}^n$ is an n-dimensional Euclidean space with the inner product $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$ and $\|x\| = \sqrt{\langle x,x\rangle}$. Let X be a set and $f\colon X \to \mathbb{R}_{+\infty}$. Then $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. A function $f\colon X \to \mathbb{R}_{+\infty}$ is called proper if $\operatorname{dom} f \neq \emptyset$. Let H be a set of functions $h\colon X \to \mathbb{R}$. A function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is called abstract convex with respect to H (H-convex) at a point $\bar{x} \in X$ if there exists a set $U \subset H$ such that $\sup\{h(x) : h \in U\} \leq f(x)$ for all $x \in X$ and $f(\bar{x}) = \sup\{h(\bar{x}) : h \in U\}$. If f is H-convex at each point $\bar{x} \in X$ then f is called H-convex on X.

Let L be a set of finite functions defined on $\mathbb{R}^n$ and $X \subset \mathbb{R}^n$. Let $f\colon \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and $x_0 \in \operatorname{dom} f$. An element $l \in L$ is called an (L,X)-subgradient of f at a point $x_0 \in X$ respect to X if

$$f(x) \geq f(x_0) + l(x) - l(x_0), \qquad \text{for each} \quad x \in X.$$

The set $\partial_{L,X} f(x)$ of all (L,X)-subgradients of f at x_0 with respect to X is referred to as (L,X)-subdifferential of f at x_0 with respect to X.

Let H_L be the set of all functions $h(x) = l(x) - c$ with $l \in L$ and $c \in \mathbb{R}$. It is easy to check that $\partial_{L,X} f(\bar{x}) \neq \emptyset$ if and only if $f(\bar{x}) = \max\{h(\bar{x}) : h \in \operatorname{supp}(f, H_L)\}$ where $\operatorname{supp}(f, H_L)$ is the set of all functions $h \in H_L$ such that $h(x) \leq f(x)$ for all $x \in X$. So the nonemptiness of L-subdifferential $\partial_{L,X} f(\bar{x})$ implies H_L-convexity of f at $\bar{x}$.

If L is the set of all linear functions defined on $\mathbb{R}^n$ and $X \subset \mathbb{R}^n$ is an open convex set then for any proper lower semicontinuous convex function $f\colon \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and

$x \in X$ we have $\partial_{L,X} f(x) = \partial f(x)$, where $\partial f(x)$ is the subdifferential of f in the sense of convex analysis.

Let $D \subset \mathbb{R}^n$ and L be a cone of functions l: $\mathbb{R}^n \to \mathbb{R}$. The *normal cone* of D with respect to L at a point $x \in D$ is given by

$$N_{L,D}(x) := \{l \in L : l(y) - l(x) \leq 0 \quad \text{for any} \quad y \in D\}.$$

It is easy to see that

$$N_{L,D}(x) = \partial_{L,X}\delta_D(x), \quad x \in D,$$

where $X = \mathbb{R}^n$ and the *indicator function* δ_D: $\mathbb{R}^n \to \mathbb{R}_{+\infty}$ is defined as

$$\delta_D(x) = \begin{cases} 0 & \text{if } x \in D, \\ +\infty & \text{if } x \notin D \end{cases}$$

and $\partial_{L,X}\delta_D(x)$ is the (L, X)-subdifferential of δ_D at x with respect to X. We know that if D is a convex set, $N_D(x) = \partial\delta_D(x)$, where $N_D(x)$ is the normal cone of set D in the sense of convex analysis. Observe that if L is the set of all linear functions defined on $\mathbb{R}^n$, D is a convex set, then

$$N_{L,D}(x) = \partial_{L,X}\delta_D(x) = \partial\delta_D(x) = N_D(x)$$

for any $x \in D$, where $X = \mathbb{R}^n$.

Definition 2.1 Let $X \subset \mathbb{R}^n$ be a convex set and let ρ be a real number. A function f: $X \to \mathbb{R}$ is said to be a ρ-convex function over X if there exists a convex function h: $X \to \mathbb{R}$ such that $f(x) = h(x) + \rho\|x\|^2$ for any $x \in X$.

If $\rho < 0$, then f is said to be a weakly convex function over X.

It is known (see [9] and references therein) that a function f is ρ-convex if and only if f is ρ-paraconvex, that is

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \rho t(1-t)\|x - y\|^2$$

for all $x, y \in \mathbb{R}^n$ and $t \in (0, 1)$. Definition 2.1 is given in Ref. [14] for $X = \mathbb{R}^n$. Obviously, if $\rho \geq 0$, then f is a convex function on X. In this paper, we consider optimization problems with weakly convex functions involved.

3 Sufficient global optimality conditions

3.1 Sufficient conditions in terms of (L, X)-subdifferentials and L-normal cones

Consider the following optimization problem (P):

$$\text{minimize} \quad g_0(x) \quad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m, \quad x \in S, \tag{3.1}$$

where $S \subset \mathbb{R}^n$ and g_i is a function defined on a set $X \supset S$, $i = 0, 1, \ldots, m$. For a given $\lambda := (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$, define

$$F_\lambda(x) := g_0(x) + \sum_{i=1}^{m} \lambda_i g_i(x), \tag{3.2}$$

$$C := \{x \in S \mid g_i(x) \leq 0, i = 1, \ldots, m\}. \tag{3.3}$$

Theorem 3.1 (Sufficient conditions for global minimizer) *Let L be a set of real-valued functions defined on $\mathbb{R}^n$ and $-l \in L$ for each $l \in L$. Let $\bar{x} \in C$. If there exists a $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$ such that $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$ and*

$$-\partial_{L,X} F_\lambda(\bar{x}) \cap N_{L,S}(\bar{x}) \neq \emptyset \tag{3.4}$$

then $\bar{x}$ is a global minimizer of problem (P).

Proof Let $x \in C$. The condition (3.4) implies that there exist $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$ and $l \in N_{L,S}(\bar{x})$ with $-l \in \partial_{L,X} F_\lambda(\bar{x})$. Then,

$$g_0(x) - g_0(\bar{x}) \geq F_\lambda(x) - F_\lambda(\bar{x}) \geq -l(x) + l(\bar{x}).$$

The inclusion $l \in N_{L,S}(\bar{x})$ implies that $l(x) - l(\bar{x}) \leq 0$. Hence, $g_0(x) - g_0(\bar{x}) \geq 0$. Since $x \in C$ is arbitrary, $\bar{x}$ is a global minimizer of problem (P). □

In Sect. 4, we will apply Theorem 3.1 to examine some problems with ρ-convex functions involved.

Corollary 3.1 *Let $g_i, i = 0, 1, \ldots, m$ be proper lower semicontinuous convex functions on $\mathbb{R}^n$, $S \subset \mathbb{R}^n$ be a convex set and let $\bar{x} \in C$. If there exists a $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$ and*

$$-\partial F_\lambda(\bar{x}) \cap N_S(\bar{x}) \neq \emptyset \tag{3.5}$$

then $\bar{x}$ is a global minimizer of problem (P).

Proof The conclusion follows from Theorem 3.1 by taking L as the set of all linear functions defined on $\mathbb{R}^n$. □

3.2 (L, X)-subdifferential of continuously differentiable functions

In this paper, we mainly consider the following set L of elementary functions:

$$L = \left\{ l \mid l(x) = \alpha \|x\|^2 + x^T \beta,\ \alpha \in \mathbb{R},\ \beta \in \mathbb{R}^n \right\}. \tag{3.6}$$

In order to apply the sufficient conditions given in previous subsection we need to calculate L-subdifferential with respect to a set X for some classes of functions.

Let f be a continuously differentiable function defined on an open convex set $X \supset S$. We begin with the description of $\partial_{L,X} f(x)$ under some assumptions.

Theorem 3.2 *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in S$, $l \in L$ with $l(x) = \alpha\|x\|^2 + \beta^T x$ and let $\varphi(x) := f(x) - l(x)$. Assume that $\varphi(x)$ is convex on X and f is continuously differentiable at $\bar{x}$. Then $l \in \partial_{L,X} f(\bar{x})$ if and only if $2\alpha\bar{x} + \beta = \nabla f(\bar{x})$.*

Proof By definition, $l \in \partial_{L,X} f(\bar{x})$ with $l(x) = \alpha\|x\|^2 + \beta^T x$ if and only if

$$l(x) - l(\bar{x}) \leq f(x) - f(\bar{x}) \quad \text{for any } x \in X, \tag{3.7}$$

i.e.,

$$\varphi(x) - \varphi(\bar{x}) \geq 0 \quad \text{for any } x \in X.$$

Thus, $l \in \partial_{L,X} f(\bar{x})$ if and only if $\bar{x}$ is a global minimizer of $\varphi(x)$ on X. If $\varphi(x)$ is convex on X and if $\varphi(x)$ is continuously differentiable at $\bar{x}$, then $\bar{x}$ is a global minimizer of $\varphi(x)$ on X if and only if $\nabla\varphi(\bar{x}) = 0$, i.e., $2\alpha\bar{x} + \beta = \nabla f(\bar{x})$. □

Remark 3.1 Let f be a continuously differentiable weakly convex function defined on X, where $X \supset S$ is an open convex set. It follows from Theorem 3.2, that $\partial_{L,X} f(\bar{x}) \neq \emptyset$ for any $\bar{x} \in S$, which implies that f is H_L-convex on S, where $H_L = \{l+c \mid l \in L, c \in \mathbb{R}\}$. But conversely, if f is H_L-convex on S, then f may not be a ρ-convex function on S. For example, let $f(x) = \begin{cases} x^3 \sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0 \end{cases}$ and let $S = \mathbb{R}$. By Ref. [11], we know that f is an H_L-convex function if and only if f is lower semicontinuous and there exists a $h \in H_L$ such that $f(x) \geq h(x)$ for all $x \in \mathbb{R}$. Obviously, f is a continuously differentiable H_L-convex function on $\mathbb{R}$, but we can easily verify that f is not a ρ-convex function on $\mathbb{R}$.

Corollary 3.2 *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in S$, $l \in L$ with $l(x) = \alpha\|x\|^2 + \beta^T x$ and let f be twice continuously differentiable on X, $H(x)$ be the Hessian matrix of f at a point x and*

$$\mu_{f,X} := \inf_{x \in X, \|y\|=1} y^T H(x) y. \tag{3.8}$$

If $\mu_{f,X} \geq 2\alpha$, then $l \in \partial_{L,X} f(\bar{x})$ if and only if $2\alpha\bar{x} + \beta = \nabla f(\bar{x})$.

Proof Let $\varphi(x) := f(x) - l(x)$. Since f is twice continuously differentiable on X and $2\alpha \leq \mu_{f,X}$, we have that $\varphi(x) = f(x) - l(x)$ is convex on X. The result follows from Theorem 3.2. □

Note that if X is a bounded set, then for each twice continuously differentiable function f on X there exists a ρ such that f is a ρ-convex function on X. Indeed, $\mu_{f,X} > -\infty$ since X is bounded.

Consider a continuously differentiable function f. Assume that the mapping $x \mapsto \nabla f(x)$ is Lipschitz continuous on X:

$$K_{f,X} := \sup_{x,y \in X, x \neq y} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|} < +\infty. \tag{3.9}$$

It was shown in Ref. [12] that such a function can be represented as the minimum of a family of functions of the form $h(x) = l(x) + c$ with $l \in L$ and $c \in \mathbb{R}$. The explicit description of this family also was given in Ref. [12]. Using this description we can characterize some elements of the (L, X)-subdifferential.

Proposition 3.1 *Let $\bar{x} \in S$ and let X be a convex open set containing S. Assume that f is a differentiable function defined on X and (3.9) holds. Let $\alpha \leq -K_{f,X}$. Then, for $l(x) = \alpha\|x\|^2 + \beta^T x$, it holds $l \in \partial_{L,X} f(\bar{x})$ if and only if $2\alpha\bar{x} + \beta = \nabla f(\bar{x})$.*

Proof For any $t \in X$, let $\alpha \leq -K_{f,X}$ and let $f_t(x) := f(t) + \langle \nabla f(t), x - t\rangle + \alpha\|x - t\|^2$ and $l_t(x) := \alpha\|x\|^2 + \langle \nabla f(t) - 2\alpha t, x\rangle$. Then,

$$f_t(x) = l_t(x) + f(t) - l_t(t).$$

First, we prove that $f(x) = \max_{t \in X} f_t(x),\ x \in X$. Applying the mean value theorem we can find for each $x, y \in X$ a number $\theta \in [0, 1]$ such that:

$$(-f(x)) - (-f(y)) = [-\nabla f(y + \theta(x - y)), x - y], \quad \theta \in [0, 1]$$

therefore

$$\begin{aligned} f(y) - f(x) + [\nabla f(y), x - y] &= [-\nabla f(y + \theta(x - y)) + \nabla f(y), x - y] \\ &\leq \|[-\nabla f(y + \theta(x - y)) + \nabla f(y)\| \|x - y\| \\ &\leq K_{f,X}\theta\|x - y\|^2 \leq K_{f,X}\|x - y\|^2 \leq -\alpha\|x - y\|^2. \end{aligned}$$

This means that

$$f(x) \geq f(y) + [\nabla f(y), x - y] + \alpha \|x - y\|^2 := f_y(x), \quad x \in X.$$

Since $f(x) = f_x(x)$ it follows that $f(x) = \max_{t \in X} f_t(x)$ for all $x \in X$. Thus, for any given $\bar{x} \in S$, we have that

$$f(x) \geq f_{\bar{x}}(x) = l_{\bar{x}}(x) + f(\bar{x}) - l_{\bar{x}}(\bar{x}), \quad \forall x \in X.$$

Thus, for $l_{\bar{x}}(x) = \alpha\|x\|^2 + \langle \nabla f(\bar{x}) - 2\alpha\bar{x}, x\rangle$, it holds $l_{\bar{x}} \in \partial_{L,X} f(\bar{x})$.

Conversely, if $l(x) := \alpha\|x\|^2 + \beta^T x \in \partial_{L,X} f(\bar{x})$, then $\bar{x}$ is a global minimizer of $f(x) - l(x)$ on X. Thus, $\bar{x}$ is a local minimizer of $f(x) - l(x)$. Hence $\nabla f(\bar{x}) - \nabla l(\bar{x}) = \nabla f(\bar{x}) - 2\alpha\bar{x} - \beta = 0$ since X is an open set. □

For any twice continuously differentiable function f we have

$$K_{f,X} \geq \mu_{f,X},$$

where $K_{f,X}$ and $\mu_{f,X}$ are defined by (3.9) and (3.8), respectively. Indeed, for any $x, y \in X, x \neq y$, there exists a $\theta \in [0, 1]$ such that

$$\begin{aligned} K_{f,X} &\geq \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|} = \frac{\|H(y + \theta(x - y))(x - y)\|}{\|x - y\|} \\ &\geq \frac{|(x - y)^T H(y + \theta(x - y))(x - y)|}{\|x - y\|^2} \geq \frac{(x - y)^T H(y + \theta(x - y))(x - y)}{\|x - y\|^2} \\ &\geq \mu_{f,X}. \end{aligned}$$

A description of L-subdifferential for quadratic functions has been given in Ref. [8].

Proposition 3.2 *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in S$, X be an open convex set containing S, $l \in L$ with $l(x) = \alpha\|x\|^2 + \beta^T x$ and let f be a quadratic function with $f(x) = \frac{1}{2}x^T Bx + b^T x + c$, where B is a symmetric matrix. Let μ_B be the minimal eigenvalue of matrix B. Then $l \in \partial_{L,X} f(\bar{x})$ if and only if $\alpha \leq \mu_B$ and $\beta = b + (B\bar{x} - 2\alpha\bar{x})$.*

3.3 Sufficient conditions for global minimizers of ρ-convex problems

In this subsection, we will give sufficient optimality conditions for some $\rho-$convex programming problems (P) defined by (3.1).

Theorem 3.3 *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C$ and let X be an open convex set such that $X \supset S$. Assume that g_0 and $g_i, i = 1, \ldots, m$ are continuously differentiable on X. For $\alpha \in \mathbb{R}$, $\beta = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$ consider the function $l_{\alpha,\beta}(x) = \alpha\|x\|^2 + \beta^T x$. If there exist $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$ and $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^n$ such that*

$$\begin{gathered} F_\lambda(x) + \alpha\|x\|^2 \text{ is convex on } X, \\ -2\alpha\bar{x} - \beta = \nabla F_\lambda(\bar{x}) \end{gathered} \tag{3.10}$$

and

$$\begin{gathered} \lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m, \\ l_{\alpha,\beta} \in N_{L,S}(\bar{x}) \end{gathered}$$

then $\bar{x}$ is a global minimizer of problem (P).

Proof By Theorem 3.2, we know that if there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ such that (3.10) holds, then $-l_{\alpha,\beta} \in \partial_{L,X} F_\lambda(\bar{x})$. If also $l_{\alpha,\beta} \in N_{L,S}(\bar{x})$ then $l \in -\partial_{L,X} F_\lambda(\bar{x}) \cap N_{L,S}(\bar{x})$. Thus, the result follows from Theorem 3.1. □

Let g_0 and g_i, $i = 0, 1, \ldots, m$ be ρ_i-convex functions on X, then there must exist ρ_λ, such as, $\rho_\lambda := \rho_0 + \sum_{i=1}^m \lambda_i \rho_i$ such that F_λ is ρ_λ-convex. In such a case there exist α and β such that (3.10) holds. (See Remark 3.1.)

We now show how to choose numbers ρ_i and ρ_λ in the case of quadratic functions. Let $g_i, i = 0, 1, \ldots, m$ be quadratic functions with $g_i(x) = x^T A_i x + a_i^T x$, $i = 0, 1, \ldots, m$, where $A_i, i = 0, 1, \ldots, m$ are symmetric matrices. For a given $\lambda \in \mathbb{R}^m_+$, let

$$A_\lambda := A_0 + \sum_{i=1}^m \lambda_i A_i. \tag{3.11}$$

Then, here we can take

$$\rho_\lambda = \mu(A_\lambda), \;\; \rho_i := \mu(A_i), \quad i = 1, \ldots, m$$

or take

$$\rho_\lambda = \mu(A_0) + \sum_{i=1}^m \lambda_i \mu(A_i),$$

where $\mu(A)$ is the minimal eigenvalue of A. Note that here $\mu(A_\lambda)$ and $\mu(A_0) + \sum_{i=1}^m \lambda_i \mu(A_i)$ may be different and generally, $\mu(A_\lambda) \geq \mu(A_0) + \sum_{i=1}^m \lambda_i \mu(A_i)$.

Consider now twice continuously differentiable functions g_i defined on X, where X is a bounded set. Let $G_i(x)$ and $H_\lambda(x)$ be the Hessian matrices of g_i and F_λ, respectively. Then we can take

$$\rho_i \leq \inf_{\|y\|=1, x \in X} y^T G_i(x) y$$

and

$$\rho_\lambda \leq \inf_{\|y\|=1, x \in X} y^T H_\lambda(x) y.$$

4 Sufficient conditions for special cases of ρ-convex problems

In this section, we give sufficient conditions for some classes of ρ-convex problems.

4.1 Problems with ρ-convex inequality constraints

Consider the following problem ($WP1$):

$$\text{minimize} \quad g_0(x) \quad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m, \quad x \in \mathbb{R}^n, \tag{4.1}$$

where $g_i, i = 0, 1, \ldots, m$ are ρ_i-convex functions on $\mathbb{R}^n$. Let $X = \mathbb{R}^n$. It was mentioned in the previous section that for each $\lambda \in \mathbb{R}^m_+$ there exists ρ_λ such that the function $F_\lambda(x) = g_0(x) + \sum_{i=1}^m \lambda_i g_i(x)$ is ρ_λ-convex.

Theorem 4.1 *Let* $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$ *be a vector such that* $\lambda_i g_i(\bar{x}) = 0$, $i = 1, \ldots, m$ *and let* ρ_λ *be a number such that* F_λ *is* ρ_λ*- convex. Let* $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C$. *Assume that* $g_i, i = 0, 1 \ldots, m$ *are continuously differentiable on* $\mathbb{R}^n$. *If* $\rho_\lambda \geq 0$ *and*

$$\nabla g_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0$$

then $\bar{x}$ *is a global minimizer of problem* $(WP1)$.

Proof Let $l(x) = -\rho_\lambda \|x\|^2 + \beta x$, where $\beta \in \mathbb{R}^n$. Then, we can verify that $l \in N_{L,\mathbb{R}^n}(\bar{x})$ if and only if

$$\rho_\lambda \geq 0 \quad \text{and } \beta = 2\rho_\lambda \bar{x}.$$

Indeed, $l \in N_{L,\mathbb{R}^n}(\bar{x})$ if and only if for any $x \in \mathbb{R}^n$,

$$-\rho_\lambda \|x\|^2 + \beta x - [-\rho_\lambda \|\bar{x}\|^2 + \beta \bar{x}] = -\rho_\lambda \|x - \bar{x}\|^2 + (\beta - 2\rho_\lambda \bar{x})(x - \bar{x}) \leq 0,$$

which is equivalent to

$$\rho_\lambda \geq 0 \quad \text{and } \beta = 2\rho_\lambda \bar{x}.$$

By definition of ρ_λ, the function $F_\lambda(x) - \rho_\lambda \|x\|^2$ is convex on $\mathbb{R}^n$. Furthermore, we can easily verify that if $\nabla g_0(\bar{x}) + \sum_{i=1}^m \lambda_i \nabla g_i(\bar{x}) = 0$, then the condition (3.10) holds. Thus, the result follows from Theorem 3.3. □

Corollary 4.1 *Let* $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C$. *Assume that* $g_i, i = 0, 1 \ldots, m$ *are continuously differentiable* ρ_i*-convex functions on* X. *Let* $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$ *be such that* $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$. *Let*

$$\alpha_\lambda := \rho_0 + \sum_{i=1}^{m} \lambda_i \rho_i. \tag{4.2}$$

If $\alpha_\lambda \geq 0$ *and*

$$\nabla g_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0.$$

Then $\bar{x}$ *is a global minimizer of problem* $(WP1)$.

It follows from Theorem 4.1 by taking $\rho_\lambda = \alpha_\lambda$.

Consider now the following problem (WEP) with both inequality and equality constraints:

$$\text{minimize } g_0(x) \text{ subject to } g_i(x) \leq 0, \quad i = 1, \ldots, m, \quad h_j(x) = 0, \quad j = 1, \ldots, k, \quad x \in \mathbb{R}^n,$$

where g_i and h_j are continuously differentiable function defined on a set X; $g_i(x)$ is a ρ_i-convex function, $h_j(x)$ is a γ_j-convex function, and $-h_j$ is a δ_j-convex function, $i = 0, 1, \ldots, m, j = 1, \ldots, k$. Let $C_E := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, h_j(x) = 0, i = 1, \ldots, m, j = 1, \ldots, k\}$.

The following assertion holds:

Proposition 4.1 *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C_E$. Suppose that $g_i, h_j, i = 0, 1 \ldots, m, j = 1, \ldots, k$ are as above and suppose that there exist $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$ and $\mu = (\mu_1, \ldots, \mu_k)^T \in \mathbb{R}^k$ such that $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$ and*

$$\nabla g_0(\bar{x}) + \sum_{i=1}^m \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^k \mu_j \nabla h_j(\bar{x}) = 0. \tag{4.3}$$

If $\alpha_{\lambda,\mu} \geq 0$, then $\bar{x}$ is a global minimizer of problem (WEP), where

$$\alpha_{\lambda,\mu} := \rho_0 + \sum_{i=1}^m \lambda_i \rho_i + \sum_{j=1}^k (\mu_j^+ \gamma_j + \mu_j^- \delta_j), \tag{4.4}$$

$\mu_j^+ := \max\{\mu_j, 0\}$ *and* $\mu_j^- := \max\{-\mu_j, 0\}$.

Proof Obviously, problem (WEP) is equivalent to the following problem ($WEP1$):

$$\begin{aligned} \text{minimize} \quad & g_0(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & h_j(x) \leq 0, \quad j = 1, \ldots, k, \\ & -h_j(x) \leq 0, \quad j = 1, \ldots, k, \\ & x \in \mathbb{R}^n, \end{aligned}$$

i.e., $\bar{x}$ is a global minimizer of problem (WEP) if and only if $\bar{x}$ is a global minimizer of problem ($WEP1$). By Corollary 4.1, we know that if there exist a $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$ and a $\mu = (\mu_1, \ldots, \mu_k)^T \in \mathbb{R}^k$ such that

$$\alpha_{\lambda,\mu} \geq 0, \quad \lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$$

and

$$\nabla g_0(\bar{x}) + \sum_{i=1}^m \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^k \mu_j^+ \nabla h_j(\bar{x}) + \sum_{j=1}^k \mu_j^- (-\nabla h_j(\bar{x})) = 0 \tag{4.5}$$

then $\bar{x}$ is a global minimizer of problem ($WEP1$). Obviously, (4.5) is equivalent to (4.3). Thus, if there exist a $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$ and a $\mu = (\mu_1, \ldots, \mu_k)^T \in \mathbb{R}^k$ such that

$$\alpha_{\lambda,\mu} \geq 0, \quad \lambda_i g_i(\bar{x}) = 0, \quad i = 1, \ldots, m$$

and (4.3) holds, then $\bar{x}$ is a global minimizer of problem (WEP). □

Note that Proposition 4.1 coincides with Theorem 5.2 in Ref. [14] and note that all the multipliers of $\alpha_{\lambda\nu}$ are nonnegative, which can assure that the function $g_0(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^k \mu_j h_j(x)$ is a $\alpha_{\lambda\nu}$-convex function.

4.2 Problems with ρ-convex inequality constraints and box constraints

Consider the following problem ($WP2$):

$$\text{minimize} \quad g_0(x) \quad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m, \quad x \in S = \prod_{i=1}^n [u_i, v_i], \tag{4.6}$$

where $g_i, i = 0, 1, \ldots, m$ are ρ_i-convex functions on X, X is an open bounded convex set such that $\mathbb{R}^n \supset X \supset S$. Let $U = (u_1, \ldots, u_n)^T, V = (v_1, \ldots, v_n)^T$. By definition of ρ_i-convexity, $h_i(x) = g_i(x) - \rho_i\|x\|^2, i = 0, 1, \ldots, m$ is a convex function on X $(i = 0, 1, \ldots, m)$. Let $C = \{x \in \prod_{i=1}^n [u_i, v_i] \mid g_i(x) \le 0, i = 1, \ldots, m\}$. For $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C$, let

$$\widetilde{x}_i := \begin{cases} 1, & \bar{x}_i \in (u_i, v_i), \\ -1, & \bar{x}_i = u_i, \\ 1, & \bar{x}_i = v_i, \end{cases} \tag{4.7}$$

$$\widetilde{X} := \mathrm{diag}(\widetilde{x}_1, \ldots, \widetilde{x}_n), \tag{4.8}$$

$$\widehat{x}_i := \begin{cases} -1, & \bar{x}_i \in (u_i, v_i), \\ -1, & \bar{x}_i = u_i, \\ 1, & \bar{x}_i = v_i, \end{cases} \tag{4.9}$$

$$\widehat{X} := \mathrm{diag}(\widehat{x}_1, \ldots, \widehat{x}_n). \tag{4.10}$$

Theorem 4.2 *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C$. Assume that $g_i, i = 0, 1, \ldots, m$ are continuously differentiable ρ_i-convex functions on X. Suppose that there exists a $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$ such that $\lambda_i g_i(\bar{x}) = 0$, $i = 1, \ldots, m$. Let ρ_λ be a number such that F_λ is ρ_λ-convex. Suppose either of the following holds:*

1. $\rho_\lambda \ge 0$ *and*

$$\widetilde{X}\nabla F_\lambda(\bar{x}) \le 0, \ \ \widehat{X}\nabla F_\lambda(\bar{x}) \le 0; \tag{4.11}$$

2. $\rho_\lambda < 0$ *and for any $i = 1, \ldots, n$, either $\bar{x}_i = u_i$ or $\bar{x}_i = v_i$ and*

$$\widetilde{X}\nabla F_\lambda(\bar{x}) \le \rho_\lambda(V - U). \tag{4.12}$$

Then, $\bar{x}$ is a global minimizer of problem (*WP*2).

Proof The function $F_\lambda(x) - \rho_\lambda\|x\|^2$ is convex on X due to the choice of ρ_λ. Let $\beta = 2\rho_\lambda\bar{x} - \nabla F_\lambda(\bar{x})$ and $l(x) = -\rho_\lambda\|x\|^2 + \beta^T x$. Let us check that if either (4.11) or (4.12) holds, then $l \in N_{L,S}(\bar{x})$. Indeed, $l \in N_{L,S}(\bar{x})$ if and only if

$$-\rho_\lambda \sum_{i=1}^n (x_i - \bar{x}_i)^2 + (\beta - 2\rho_\lambda\bar{x})(x - \bar{x}) \le 0 \quad \text{for each } x \in S. \tag{4.13}$$

Since $S = \prod_{i=1}^n [u_i, v_i]$, it follows that (4.13) is equivalent to

$$-\rho_\lambda(x_i - \bar{x}_i)^2 + (\beta_i - 2\rho_\lambda\bar{x}_i)(x_i - \bar{x}_i) \le 0 \quad \text{for any } x_i \in [u_i, v_i], \quad i = 1, \ldots, n. \tag{4.14}$$

1. Let $\rho_\lambda \ge 0$. Then (4.14) is equivalent to the following condition:

$$\begin{array}{lll} \text{(a)} & \beta_i - 2\rho_\lambda\bar{x}_i = -(\nabla F_\lambda(\bar{x}))_i = 0 & \text{if } \bar{x}_i \in (u_i, v_i), \\ \text{(b)} & \beta_i - 2\rho_\lambda\bar{x}_i = -(\nabla F_\lambda(\bar{x}))_i \le 0 & \text{if } \bar{x}_i = u_i, \\ \text{(c)} & \beta_i - 2\rho_\lambda\bar{x}_i = -(\nabla F_\lambda(\bar{x}))_i \ge 0 & \text{if } \bar{x}_i = v_i. \end{array} \tag{4.15}$$

Indeed,

$$\rho_\lambda(x_i - \bar{x}_i)^2 - (\beta_i - 2\rho_\lambda\bar{x}_i)(x_i - \bar{x}_i) \ge 0 \qquad \text{for any} \quad x_i \in [u_i, v_i]$$

if and only if $\bar{x}_i$ is a global minimizer of convex function $r_i(x_i) = \rho_\lambda(x_i - \bar{x}_i)^2 - (\beta_i - 2\rho_\lambda\bar{x}_i)(x_i - \bar{x}_i)$ on $[u_i, v_i]$. Consider separately three cases: $\bar{x}_i \in (u_i, v_i)$, $\bar{x}_i = u_i$, $\bar{x}_i = v_i$.

(a) If $\bar{x}_i \in (u_i, v_i)$, then $r_i'(\bar{x}_i) = \beta_i - 2\rho_\lambda \bar{x}_i = 0$;
(b) Let $\bar{x}_i = u_i$. Then

$$\rho_\lambda(x_i - u_i)^2 - (\beta_i - 2\rho_\lambda u_i)(x_i - u_i) \geq 0 \quad \text{for any } x_i \in [u_i, v_i]$$

if and only if $-\rho_\lambda(x_i - u_i) + (\beta_i - 2\rho_\lambda u_i) \leq 0$ for any $x_i \in (u_i, v_i]$, i.e., $\beta_i - 2\rho_\lambda u_i \leq 0$;
(c) Let $\bar{x}_i = v_i$. Then

$$\rho_\lambda(x_i - v_i)^2 - (\beta_i - 2\rho_\lambda v_i)(x_i - v_i) \geq 0 \quad \text{for any } x_i \in [u_i, v_i]$$

if and only if $-\rho_\lambda(x_i - v_i) + \beta_i - 2\rho_\lambda v_i \geq 0$ for any $x_i \in [u_i, v_i)$, i.e., $\beta_i - 2\rho_\lambda v_i \geq 0$.

We have shown that (4.14) can be represented as (4.15). An easy calculation shows that (4.15) implies (4.11). Conversely, by (4.7)–(4.10), we know that (4.11) also implies (4.15).

2. Let $\rho_\lambda < 0$. We will prove that condition (4.14) holds if and only if either $\bar{x}_i = u_i$ or $\bar{x}_i = v_i$ and

$$\begin{array}{lll} \text{(a)} & -\rho_\lambda(v_i - u_i) - (\nabla F_\lambda(\bar{x}))_i \leq 0 & \text{if } \bar{x}_i = u_i, \\ \text{(b)} & -\rho_\lambda(u_i - v_i) - (\nabla F_\lambda(\bar{x}))_i \geq 0 & \text{if } \bar{x}_i = v_i. \end{array} \tag{4.16}$$

Indeed, if there exists $1 \leq i \leq n$ such that $u_i < \bar{x}_i < v_i$, then by (4.14), $(\beta_i - 2\rho_\lambda \bar{x}_i)(x_i - \bar{x}_i) < 0$ for any $x_i \in [u_i, v_i] \backslash \{\bar{x}_i\}$ since $-\rho_\lambda(x_i - \bar{x}_i)^2 > 0$ for any $x_i \in [u_i, v_i] \setminus \{\bar{x}_i\}$. This is impossible.

If $\bar{x}_i = u_i$, then

$$\rho_\lambda(x_i - \bar{x}_i)^2 - (\beta_i - 2\rho_\lambda \bar{x}_i)(x_i - \bar{x}_i) \geq 0 \quad \text{for any } x_i \in [u_i, v_i]$$

if and only if

$$-\rho_\lambda(v_i - u_i) + (\beta_i - 2\rho_\lambda \bar{x}_i) = -\rho_\lambda(v_i - u_i) - (\nabla F_\lambda(\bar{x}))_i \leq 0;$$

If $\bar{x}_i = v_i$, then

$$\rho_\lambda(x_i - \bar{x}_i)^2 - (\beta_i - 2\rho_\lambda \bar{x}_i)(x_i - \bar{x}_i) \geq 0 \quad \text{for any } x_i \in [u_i, v_i]$$

if and only if

$$-\rho_\lambda(u_i - v_i) + (\beta_i - 2\rho_\lambda \bar{x}_i) = -\rho_\lambda(u_i - v_i) - (\nabla F_\lambda(\bar{x}))_i \geq 0.$$

Thus, (4.14) implies (4.16). The converse implication can easily be verified. An easy calculation shows that (4.16) is equivalent to (4.12). The desired result follows from Theorem 3.3. □

Corollary 4.2 *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C$. Suppose that $g_i, i = 0, 1 \ldots, m$ are ρ_i- convex continuously differentiable on X functions and suppose that there exists $\lambda \in \mathbb{R}^m_+$ such that $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$. Let α_λ be defined by (4.2). If either of the following holds:*

1. $\alpha_\lambda \geq 0$ *and*

$$\widetilde{X}\nabla F_\lambda(\bar{x}) \leq 0, \qquad \widehat{X}\nabla F_\lambda(\bar{x}) \leq 0; \tag{4.17}$$

2. $\alpha_\lambda < 0$; *for any* $i = 1, \ldots, n$, *either* $\bar{x}_i = u_i$ *or* $\bar{x}_i = v_i$, *and*

$$\widetilde{X} \nabla F_\lambda(\bar{x}) \leq \alpha_\lambda (V - U), \tag{4.18}$$

then, $\bar{x}$ *is a global minimizer of problem* $(WP2)$.

Proof It follows from Theorem 4.2 by letting $\rho_\lambda = \alpha_\lambda$. □

4.3 Bivalent problems with ρ-convex inequality constraints

Consider the following bivalent problem $(WP3)$:

$$\text{minimize} \quad g_0(x) \quad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m, \quad x \in S = \prod_1^n \{-1, 1\},$$

where $g_i, i = 0, 1, \ldots, m$ is a continuously differentiable ρ_i-convex functions on X, X is an open convex bounded set such that $X \supset \prod_1^n [-1, 1]$. Let $C_B := \{x \in \prod_1^n \{-1, 1\} \mid g_i(x) \leq 0, i = 1, \ldots, m\}$.

Theorem 4.3 *Let* $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C_B$, $\bar{X} = \mathrm{diag}(\bar{x})$. *Suppose that there exists* $\lambda \in \mathbb{R}^m_+$ *such that* $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$. *Let* ρ_λ *be a number such that* F_λ *is* ρ_λ-*convex. If*

$$\bar{X} \nabla F_\lambda(\bar{x}) \leq 2\rho_\lambda \mathbf{1}, \qquad \textit{where} \quad \mathbf{1} = (1, \ldots, 1) \tag{4.19}$$

then $\bar{x}$ *is a global minimizer of problem* $(WP3)$.

Proof Let $l(x) = -\rho_\lambda \|x\|^2 + \beta^T x$. Then, we can verify that $l \in N_{L,S}(\bar{x})$ if and only if $\bar{X}\beta \geq 0$. Indeed, $l \in N_{L,S}(\bar{x})$ if and only if

$$-\rho_\lambda \|x - \bar{x}\|^2 + (\beta - 2\rho_\lambda \bar{x})(x - \bar{x}) \leq 0 \quad \text{for any } x \in \prod_1^n \{-1, 1\},$$

which is equivalent to

$$-\rho_\lambda (x_i - \bar{x}_i)^2 + (\beta_i - 2\rho_\lambda \bar{x}_i)(x_i - \bar{x}_i) \leq 0 \quad \text{for any } x_i \in \{-1, 1\}, \quad i = 1, \ldots, n.$$

As $x_i - \bar{x}_i = -2\bar{x}_i$ if $x_i \neq \bar{x}_i$, we know that $l \in N_{L,S}(\bar{x})$ if and only if

$$\beta_i \bar{x}_i \geq 0, \qquad \text{for any } i = 1, \ldots, n,$$

i.e., $\bar{X}\beta \geq 0$.

Furthermore, if $\beta = 2\rho_\lambda \bar{x} - \nabla F_\lambda(\bar{x})$, then condition (3.10) holds. Thus, if condition (4.19) hods, then for $l(x) = -\rho_\lambda \|x\|^2 + (2\rho_\lambda \bar{x} - \nabla F_\lambda(\bar{x}))^T x$, condition (3.10) holds and $l \in N_{L,S}(\bar{x})$. Hence the result follows from Theorem 3.3. □

Corollary 4.3 *Let* $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C_B$, $\bar{X} = \mathrm{diag}(\bar{x})$. *Suppose that there exist* $\lambda \in \mathbb{R}^m_+$ *such that* $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$. *Let* α_λ *be defined by (4.2). If*

$$\bar{X} \nabla F_\lambda(\bar{x}) \leq 2\alpha_\lambda \mathbf{1} \tag{4.20}$$

then $\bar{x}$ *is a global minimizer of problem* $(WP3)$.

Proof It follows from Theorem 4.3 by taking $\rho_\lambda = \alpha_\lambda$. □

Consider the relaxed problem ($RWP3$) of bivalent problem ($WP3$):

$$\text{minimize} \quad g_0(x) \quad \text{subject to} \quad g_i(x) \le 0, \quad i = 1, \ldots, m, \quad x \in \prod_1^n [-1, 1].$$

Corollary 4.4 *Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in C_B$, $\bar{X} = \mathrm{diag}(\bar{x})$. Suppose that there exist $\lambda \in \mathbb{R}_+^m$ such that $\lambda_i g_i(\bar{x}) = 0, i = 1, \ldots, m$ and suppose either of the following holds:*

1. $\rho_\lambda \ge 0$ *and*

$$\bar{X} \nabla F_\lambda(\bar{x}) \le 0; \tag{4.21}$$

2. $\rho_\lambda < 0$ *and*

$$\bar{X} \nabla F_\lambda(\bar{x}) \le 2\rho_\lambda \mathbf{1}. \tag{4.22}$$

Then $\bar{x}$ is a global minimizer of both problem ($WP3$) *and problem* ($RWP3$).

Proof It follows from Theorem 4.3 and Theorem 4.2. □

Example 4.1 Consider the following problem:

$$(EP2) \qquad \begin{aligned} \min \quad & g_0(x) = -\frac{1}{3}x_1^3 + x_2^2 + x_2 x_3 + x_3^2 + \sin x_4 + 2x_1 - 4x_2 + 3x_3 - 4x_4, \\ s.t. \quad & g_1(x) = 2x_1 + 2x_2 + x_3 + x_4 \le 0, \\ & g_2(x) = 3x_1 - x_2 + 2x_3 - 4x_4 + 2 \le 0, \\ & -1 \le x_i \le 1, i = 1, 2, 3, 4. \end{aligned}$$

Let $C = \{x \in \prod_1^4 [-1, 1] \mid g_i(x) \le 0, i = 1, 2\}$ and let $\bar{x} = (-1, 1, -1, 1)^T$. Obviously, $\bar{x} \in C$, $g_1(\bar{x}) = 0, g_2(\bar{x}) < 0$. Let $\rho_0 := \min_{x \in \prod_1^4[-1,1], \|y\|=1} y^T H_0(x) y$, where $H_0(x)$ is the Hessian matrix of g_0 at point x. An easy calculation shows that $\rho_0 = -2$. Let also $\rho_1 = \rho_2 = 0$. Then g_i is a ρ_i-convex function ($i = 0, 1, 2$). For any $\lambda = (\lambda_1, \lambda_2)^T \in \mathbb{R}_+^2$ such that $\lambda_i g_i(\bar{x}) = 0, i = 1, 2$, we have that $\lambda_2 = 0$ and $\alpha_\lambda = -2$. Then, we can easily verify that for $\lambda = (\frac{1}{2}, 0)^T$, we have that

$$\widetilde{X} \nabla F_\lambda(\bar{x}) = \left(-2, -2, -\frac{5}{2}, -\frac{7}{2} + \cos 1\right)^T \le (-2, -2, -2, -2)^T = \alpha_\lambda \mathbf{1},$$

i.e., condition (4.22) holds. Thus, it follows from Corollary 4.4, that $\bar{x}$ is a global minimizer of problem ($EP2$).

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